Worked out problems
Exam Complex Analysis
January 28, 2014

1.) a) Since $au(x,y)+bv(x,y)=c$ for all $x+iy \in D$ we also have that

$$\frac{\partial}{\partial x}\Big(au(x,y)+bv(x,y)\Big)=0$$

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$$\frac{\partial}{\partial y}\Big(au(x,y)+bv(x,y)\Big)=0$$

on D. This implies

$$a\frac{\partial u}{\partial x}+b\frac{\partial v}{\partial x}=0$$

$$a\frac{\partial u}{\partial y}+b\frac{\partial v}{\partial y}=0$$

on D

b) Since $f(z)$ is analytic, the Cauchy-Riemann equations hold : $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

Thus we obtain

$$a\frac{\partial v}{\partial y}+b\frac{\partial v}{\partial x}=0 \quad \text{and} \quad -a\frac{\partial v}{\partial x}+b\frac{\partial v}{\partial y}=0$$

(*)

This yields

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$$ab\frac{\partial v}{\partial y}+b^2\frac{\partial v}{\partial x}=0 \quad \text{and} \quad -a^2\frac{\partial v}{\partial x}+ab\frac{\partial v}{\partial y}=0$$

Subtracting these equations yields

$$(b^2+a^2)\frac{\partial v}{\partial x}=0$$

which implies $\frac{\partial v}{\partial x}=0$. It also follows from (*) that $a^2\frac{\partial v}{\partial y}+ab\frac{\partial v}{\partial x}=0$ and $-ab\frac{\partial v}{\partial x}+b^2\frac{\partial v}{\partial y}=0$

This implies (by adding the equations) that

$$(a^2+b^2)\frac{\partial v}{\partial y}=0 \quad \text{so} \quad \frac{\partial v}{\partial y}=0.$$

Finally, $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=0$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}=0$.

c) Obviously, $u(x,y)$ and $v(x,y)$ are therefore constant, so $f(z)$ is constant on D.

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2.) $f(z) = \sin z$

a) By definition, $\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$. Hence

4 $f(z)=0 \iff e^{iz} = e^{-iz} \iff e^{2iz} = 1 = e^{2k\pi i} \quad (k \in \mathbb{Z})$

Hence we see that the zeroes are given by

$z = k\pi \ ; \ k \in \mathbb{Z}$

b) Using that $e^{iz} = e^{i(x+iy)} = e^{-y}(\cos x + i \sin x)$ and $e^{-iz} = e^{-i(x+iy)} = e^{y}(\cos x - i \sin x)$ we can compute

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$$f(z) = \underbrace{\left(\frac{e^{-y}+e^{y}}{2}\right)\sin x}_{u(x,y)} + i \underbrace{\left(\frac{e^{y}-e^{-y}}{2}\right)\cos x}_{v(x,y)}$$

c) It is easily verified that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ on $\mathbb{C}$. Since the partial derivatives are also

4 continuous, we can conclude that $f(z)$ is analytic on $\mathbb{C}$

d) If $f(z)$ were bounded, then by Liouville's theorem it must be constant. Clearly

4 $f(z)$ is not constant, so $f(z)$ is <u>not</u> bounded

3.) a) Define $g(z) := \frac{f(z)}{z - z_0}$. Since z_0 lies <u>outside</u> the area bounded by the contour Γ, the

8 function $g(z)$ is analytic inside and on Γ

Hence $\int_\Gamma g(z)\, dz = 0$ by Cauchy's theorem

b) By Cauchy's Integral formula we have for all z inside Γ:

$$f(z) = \int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

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$$g(z) = \int_\Gamma \frac{g(\zeta)}{\zeta - z}\, d\zeta$$

Since $f(\zeta) = g(\zeta)$ on Γ, these integrals are the same for each such z, so $f(z) = g(z)$

4.) $g(z) = \dfrac{e^{-z}}{(z+1)^2}$

a) Isolated singularity at $z=-1$. Hence $g(z)$ has a Laurent series in $|z+1|>0$. First note that

$$e^{-z} = e \cdot e^{-(z+1)}.$$

$$e^{w} = 1 + w + \frac{1}{2!}w^2 + \frac{1}{3!}w^3 + \cdots\cdots$$

Hence

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$$\frac{1}{(z+1)^2}e^{-z} = \frac{e}{(z+1)^2}e^{-(z+1)}$$

$$= \frac{e}{(z+1)^2}\left\{1 - (z+1) + \frac{1}{2!}(z+1)^2 - \frac{1}{3!}(z+1)^3 + \cdots\cdots\right\}$$

$$= \frac{e}{(z+1)^2} - \frac{e}{z+1} + \frac{e}{2!} - \frac{e}{3!}(z+1) + \cdots\cdots.$$

b) From the Laurent series we see that $z=-1$ is

4 a <u>pole</u> of <u>order 2</u>

c) By the residue theorem we have

$$\int_\Gamma g(z)\,dz = 2\pi i \operatorname{Res}(g,-1)$$

where $\operatorname{Res}(g,-1)$ is the residue of $g(z)$ at -1. From the Laurent series we see that

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$$\operatorname{Res}(g,-1) = -e$$

Hence $\int_\Gamma g(z)\,dz = -2\pi i e$

5) $f(z) = z\cos\left(\frac{1}{2z}\right)$

a) $\cos w = 1 + \frac{w^2}{2!} - \frac{w^4}{4!} + \frac{w^6}{6!} - \dots$

8 $$z\cos\left(\frac{1}{2z}\right) = z\left(1 + \frac{1}{2!}\frac{1}{4z^2} - \frac{1}{4!}\frac{1}{16z^4} + \frac{1}{6!}\frac{1}{64z^6}\cdots\right)$$

$$= z + \frac{1}{2!\cdot 4}\frac{1}{z} - \frac{1}{4!\,16}\frac{1}{z^3} + \frac{1}{6!\,64}\frac{1}{z^5} - \dots$$

b) $f(z)$ has an isolated singularity at $z=0$.
Since there are infinitely many terms $\frac{1}{z^k}$ in
4 the Laurent expansion, this is
an essential singularity

c) $\mathrm{Res}(f, 0) = \frac{1}{2!\,4} = \frac{1}{8}$.
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6) a) Let $f(z)$ and $h(z)$ be analytic inside and on a simple closed contour C and assume that

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$$|h(z)| < |f(z)|$$

for all z on C. Then f and $f+h$ have the same number of zeroes inside C

b) Consider $g(z) = 6z^4 + z^3 - 2z^2 + z - \frac{7}{4}$. Define $f(z) = 6z^4$ and $h(z) = z^3 - 2z^2 + z - \frac{7}{4}$. Consider C given by $|z| = 1$. For z on C we have

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$$|f(z)| = 6|z|^4 = 6$$

$$|h(z)| \leq |z^3| + |2z^2| + |z| + \frac{7}{4}$$

$$= |z|^3 + 2|z|^2 + |z| + \frac{7}{4}$$

$$= 1 + 2 + 1 + \frac{7}{4} = 5\frac{3}{4}$$

We see that $|h(z)| \leq 5\frac{3}{4} < 6 = |f(z)|$ for $|z| = 1$. Hence $g(z)$ has the same number of zeroes in $|z| < 1$ as $f(z)$, namely 4.

c) Consider the contour $|z| = \frac{1}{2}$. Define $f(z) = -\frac{7}{4}$. Clearly it has no zeroes in $|z| < \frac{1}{2}$. Define $h(z) := 6z^4 + z^3 - 2z^2 + z$. On $|z| = \frac{1}{2}$ we have

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$$|h(z)| \leq 6|z|^4 + |z|^3 + 2|z|^2 + |z|$$

$$= 6\frac{1}{16} + \frac{1}{8} + \frac{2}{4} + \frac{1}{2} = \frac{3}{2} < \frac{7}{4} = |f(z)|$$

Hence $g(z)$ has <u>no</u> zeroes in $|z| < \frac{1}{2}$.